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The variety of positive superdivisors of a supercurve (supervortices) *

J.A. Domínguez Pérez, D. Hernández Ruipérez and C. Sancho de Salas
Departamento de Matemática Pura y Aplicada, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain [1]

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The supersymmetric product of a supercurve is constructed with the aid of a theorem of algebraic invariants and the notion of positive relative superdivisor (supervortex) is introduced. A supercurve of positive superdivisors of degree 1 (supervortices of vortex number 1) of the original supercurve is constructed as its supercurve of conjugate fermions, as well as the supervariety of relative positive superdivisors of degree p (supervortices of vortex number p). A universal superdivisor is defined and it is proved that every positive relative superdivisor can be obtained in a unique way as a pull-back of the universal superdivisor. The case of SUSY-curves is discussed.

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1. Introduction

Positive divisors of degree p on an algebraic curve X can be thought of as unordered sets of p points of X, hence, as elements of the symmetric p-fold product S^pX. The symmetric p-fold product is the orbit space of the cartesian p-fold product X^p under the natural action of the symmetric group, and it is thus endowed with a natural structure of algebraic variety. In this way, positive divisors of degree p are the points of an algebraic variety $\mathrm{Div}^p(X)$, and this variety is of great importance in the study of the geometry of curves, and it also has a growing interest in Mathematical Physics.

From the geometrical side, one has, for instance, the role played by the variety of positive divisors of degree p in some classical constructions of the Jacobian

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[1] E-mail: sanz@relay.rediris.es (subject: To D.H.Ruiperez).

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variety of a complete smooth algebraic curve. The first construction of the Jacobian variety, due to Jacobi and Abel, is of an analytic nature and defines the Jacobian as a complex torus through the period matrix. The first algebraic construction is due to Weil [31], who showed that the algebraic structure and the group law of the Jacobian come from the fact that it is birationally equivalent to the variety of positive divisors of degree the genus of the curve. Another procedure stems from Chow [7], who took advantage of the fact that for p high enough the Abel map (that maps a divisor of degree p into its linear equivalence class) is a projective bundle, to endow the Jacobian with a structure of projective algebraic group. But regardless of the method used for constructing the Jacobian, the structure of the variety of positive divisors of degree p and the diverse Abel morphisms from these varieties to the Jacobian, turn out to be a key point in the theory of Jacobian varieties (see, for instance, refs. [17,25]) and has proved to be an important tool in the solution of the Schottky problem [26].

From the physical point of view, the variety of positive divisors of a complex complete smooth curve X (a compact Riemann surface) is the variety of *vortices* or solutions to the vortex equations [5,10]. For every holomorphic line bundle L on X endowed with a hermitian metric, there is a Yang–Mills–Higgs functional $\mathrm{YMH}_\tau(\nabla,\phi)$ defined on gauge equivalence classes of pairs (∇,ϕ) where ∇ is a unitary connection, by

$$\mathrm{YMH}_\tau(\nabla,\phi) = \int (|F_\nabla|^2 + |\nabla\phi|^2 + \tfrac{1}{4}|\phi\otimes\phi* - \tau\,\mathrm{Id}|^2)\,\mathrm{d}\mu\,,$$

where F_∇ is the curvature, $\nabla\phi$ the covariant derivative, and τ is a real parameter (see ref. [5]).

Bradlow's theorem states that for large τ, gauge equivalence classes of solutions (∇,ϕ) to the vortex equation

$$\mathrm{YMH}_\tau(\nabla,\phi) = 2\pi p\tau\,,$$

where p is the degree of L with respect to the Kähler form, correspond to divisors of degree p on X. In this correspondence, a solution (∇,ϕ) corresponds to the divisor given by the set of centres of the vortices appearing with multiplicity given by the multiplicity of the magnetic flux.

There is no similar theory for supersymmetric extensions of the vortex equations (supervortices), and in fact only very little work on supervortices or supersymmetric extensions of the Bogomolny equations has been done (see ref. [20]). This paper will provide a first step in that direction, by providing the right supervariety of positive superdivisors or supervortices on a supercurve.

This paper is organized as follows:

The *supersymmetric product* $S^p\mathcal{X}$ for a supercurve $\mathcal{X}$ of dimension $(1,1)$ is constructed in section 2 as the orbit ringed space obtained through the action of the symmetric group on the cartesian p-fold product of $\mathcal{X}$. It is far from trivial that the resulting graded ringed space is a supervariety of dimension (p,p), a statement which is shown to be equivalent to an invariant theorem. It should be stressed that this theorem is no longer true for supercurves of higher odd dimension, but our result covers the most important cases such as SUSY-curves.

In section 3 the notion of *positive relative superdivisor* of degree p for a relative supercurve $\mathcal{X} \times \mathcal{S} \to \mathcal{S}$ is given. The classical definition cannot be extended straightforwardly to supercurves if we wish that superdivisors could be obtained as pull-backs of a suitable universal superdivisor.

For ordinary algebraic curves, positive divisors of degree 1 are just points. The novelty here is that for an algebraic supercurve $\mathcal{X}$, positive relative superdivisors of degree 1 (supervortices of vortex number 1) are *are not* points of $\mathcal{X}$ (see ref. [23]), but rather they are points of another supercurve $\mathcal{X}^c$ with the same underlying ordinary (bosonic) curve. Actually, if we think of $\mathcal{X}$ as a field of fermions on a bosonic curve, the supercurve $\mathcal{X}^c$ is the supercurve of conjugate fermions on the underlying bosonic curve.

This is proven in section 4, which also contains the representability theorem for positive relative superdivisors of degree p on a supercurve. The theorem means that positive relative superdivisors of degree p are the points of the supersymmetric p-fold product $S^p\mathcal{X}^c$ of the supercurve $\mathcal{X}^c$ of conjugate fermions. This property is stated in the spirit of Algebraic Geometry in terms of the functor of the points; the precise statement is that the functor of the positive relative superdivisors of degree p of a supercurve $\mathcal{X}$ of odd dimension 1, is the functor of the points of the supersymmetric p-fold product $S^p\mathcal{X}^c$. This means that every positive relative superdivisor of $\mathcal{X} \times \mathcal{S} \to \mathcal{S}$ can be obtained in a unique way as the pull-back of a certain universal positive superdivisor through a morphism $\mathcal{S} \to S^p\mathcal{X}^c$. We obtain in that way what is the right structure of algebraic superscheme the "space" of positive superdivisors of degree p on a supercurve can be endowed with.

The case of supersymmetric curves (SUSY-curves) is particularly important, both for historical and geometrical reasons. We prove that for a supercurve $\mathcal{X}$, the existence of a conformal structure on $\mathcal{X}$ is equivalent to the existence of an isomorphism between $\mathcal{X}$ and the supercurve $\mathcal{X}^c$ of conjugate fermions. In other words, a supercurve $\mathcal{X}$ is a SUSY-curve if and only if $\mathcal{X}$ is isomorphic with $\mathcal{X}^c$. In this case the universal positive superdivisor of degree 1 is Manin's superdiagonal [4,23] and we recover from a clearer and more general viewpoint Manin's interpretation of the relationship between points and positive divisors of degree 1 for SUSY-curves, and some connected definitions [28,29].

Summarizing, the space of supervortices of vortex number p (positive superdivisors of degree p) on a supercurve $\mathcal{X}$ of odd dimension 1, is an algebraic supervariety of dimension (p,p). This algebraic supervariety is the supervariety $S^p\mathcal{X}^c$ of "unordered families" of p conjugate fermions. Moreover, only for SUSY-curves are supervortices of vortex number p "unordered families" of p points of $\mathcal{X}$.

This theory can be extended straightforwardly to SUSY-families parametrized by an *ordinary* algebraic scheme.

The results of this paper for the case of SUSY-curves only were stated (without proofs) in ref. [8].

2. Supersymmetric products

2.1. DEFINITIONS

A suitable reference for schemes theory is ref. [14]; the general theory of schemes in supergeometry (superschemes) can be found in refs. [22] and [27].

Let $\mathcal{X} = (X, \mathcal{A})$ be a graded ringed space, that is, a pair consisting of a topological space X endowed with a sheaf $\mathcal{A}$ of $\mathbb{Z}_2$-graded algebras. Let us denote by $\mathcal{J}$ the ideal $\mathcal{A}_1 + \mathcal{A}_1^2$.

Definition 1. A superscheme of dimension (m, n) over a field k, is a graded ringed space $\mathcal{X} = (X, \mathcal{A})$ where $\mathcal{A}$ is a sheaf of graded k-algebras such that:

(i) $(X, \mathcal{O} = \mathcal{A}/\mathcal{J})$ is an m-dimensional scheme of finite type over k;

(ii) $\mathcal{J}/\mathcal{J}^2$ is a locally free $\mathcal{O}$-module of rank n and $\mathcal{A}$ is locally isomorphic to $\bigwedge_{\mathcal{O}}(\mathcal{J}/\mathcal{J}^2)$.

Definition 2. A superscheme $\mathcal{X} = (X, \mathcal{A})$ over a field k is said to be affine if the underlying scheme $(X, \mathcal{O} = \mathcal{A}/\mathcal{J})$ is an affine scheme, that is, if there is a homeomorphism

$$X \xrightarrow{\sim} \operatorname{Spec}(\Gamma(X, \mathcal{O}))$$

and $\mathcal{O}$ is the sheaf on X defined by localization on the basic open subsets of the spectrum.

If $\mathcal{X} = (X, \mathcal{A})$ is an affine superscheme, and $A = \Gamma(X, \mathcal{A})$, we shall simply write $\mathcal{X} = \operatorname{Spec} A$ for it.

Let us consider the product

$$\mathcal{X}^g = (X^g, \mathcal{A}^{\otimes g}),$$

where X^g denotes the cartesian product $X \times \overset{g)}{\cdots} \times X$, and $\mathcal{A}^{\otimes g} = \mathcal{A} \otimes \overset{g)}{\cdots} \otimes \mathcal{A}$.

The symmetric group S_g acts on $\mathcal{X}^g$ by graded automorphisms of superschemes according to the rule

$$\sigma\colon X^g \to X^g\,,$$
$$(x_1,\dots,x_g)\mapsto (x_{\sigma(1)},\dots,x_{\sigma(g)})\,,$$
$$\sigma^*\colon \mathcal{A}^{\otimes g}\to \sigma_*\mathcal{A}^{\otimes g}\,,$$
$$f_1\otimes\cdots\otimes f_g \mapsto \prod_{\substack{i<j\\ \sigma(i)>\sigma(j)}} (-1)^{|f_i||f_j|} f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(g)}\,, \tag{1}$$

where $|\ |$ stands for the $\mathbb{Z}_2$-degree. This action reduces to the ordinary action of S_g on the scheme $(X^g, \mathcal{O}^{\otimes g})$. Then, we have the orbit space $S^g X$, a natural projection $p\colon X^g \to S^g X$, and an invariant sheaf $\mathcal{O}_g = \mathcal{O}^{S_g}$ on $S^g X$, whose sections on an open subset $V \subseteq S^g X$ are

$$\mathcal{O}_g(V) = \{f \in \mathcal{O}^{\otimes g}(p^{-1}(V))\,|\,\sigma^* f = f \text{ for every } \sigma \in S_g\}\,.$$

It is well known that, if $(X, \mathcal{O})$ is a projective scheme, the ringed space $(S^g X, \mathcal{O}_g)$ is a scheme, the *symmetric p-fold product* of $(X, \mathcal{O})$ [30, prop. 19].

Let us consider the sheaf $\mathcal{A}_g = (\mathcal{A}^{\otimes g})^{S_g}$ of graded invariants on $S^g X$ defined as above by letting

$$\mathcal{A}_g(V) = \{f \in \mathcal{A}^{\otimes g}(p^{-1}(V))\,|\,\sigma^* f = f \text{ for every } \sigma \in S_g\}$$

for every open subset $V \subseteq S^g X$.

2.2. THE CASE OF SUPERCURVES

Definition 3. A supercurve is a superscheme $\mathcal{X}$ of dimension $(1,n)$ over a field k.

Let $\mathcal{X}$ be a smooth proper supercurve, that is, a supercurve such that $(X, \mathcal{O})$ is proper and smooth.

Theorem 1. *Let $\mathcal{X} = (X, \mathcal{A})$ be a smooth proper supercurve of odd dimension $n > 0$. The graded ringed space $S^g \mathcal{X}$ is a superscheme if and only if $n = 1$, that is, if and only if $\mathcal{X}$ is a superscheme of dimension $(1,1)$. In that case, $S^g \mathcal{X} = (S^g X, \mathcal{A}_g)$ is a superscheme of dimension (g,g), which will be called the supersymmetric g-fold product of $\mathcal{X}$.*

Proof. Let us notice that $(X, \mathcal{O})$ is projective (it has very ample sheaves), so that the ringed space $(S^g X, \mathcal{O}_g)$ is a scheme as we mentioned above (in fact, it is smooth, which is no longer true for higher dimensional X).

As there is a natural projection $\mathcal{A}_g \to \mathcal{O}_g$, we only have to ascertain if $\mathcal{A}_g$ is locally the exterior algebra of a locally free $\mathcal{O}_g$-module. We can thus assume $\mathcal{A} = \wedge_{\mathcal{O}}(\mathcal{N})$, $\mathcal{N}$ being a free rank-n $\mathcal{O}$-module.

Let us write

$$\mathcal{N}_i = \mathcal{O} \otimes \cdots \otimes \overset{\downarrow i)}{\mathcal{N}} \otimes \cdots \otimes \mathcal{O}$$

and $\mathcal{M} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_g$. Now, if $\overline{\mathcal{O}} = \mathcal{O}^{\otimes g}$ and $\overline{\mathcal{A}} = \mathcal{A}^{\otimes g}$, we have

$$\overline{\mathcal{A}} = \Lambda_{\mathcal{O}}(\mathcal{N}) \otimes_{\mathcal{O}} \overset{g)}{\cdots} \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}(\mathcal{N}) \simeq \Lambda_{\overline{\mathcal{O}}}(\mathcal{M}) .$$

The symmetric group S_g acts on $\mathcal{M}$ by

$$\sigma(n_1 + \cdots + n_g) = n_{\sigma(1)} + \cdots + n_{\sigma(g)}$$

and this action provides an action $\sigma\colon \overline{\mathcal{A}} \to \overline{\mathcal{A}}$ on the exterior algebra $\overline{\mathcal{A}} = \Lambda_{\overline{\mathcal{O}}}(\mathcal{M})$, given by

$$\sigma(m_1 \wedge \cdots \wedge m_p) = \sigma(m_1) \wedge \cdots \wedge \sigma(m_p) .$$

This action of S_g on $\overline{\mathcal{A}}$ is actually equal to the one defined in (1), because both coincide on $\Lambda^1_{\overline{\mathcal{O}}}(\mathcal{M}) = \mathcal{M}$ and are morphisms of graded algebras.

If we denote by $\mathcal{M}^{S_g}$ the $\mathcal{O}_g$-module consisting of the invariant sections of $\mathcal{M}$, the proof of theorem 1 will be thus completed with the following

Lemma 1. *The natural morphism of sheaves of graded $\mathcal{O}_g$-algebras over $S^g X$,*

$$\phi\colon \Lambda_{\mathcal{O}_g}(\mathcal{M}^{S_g}) \to (\Lambda_{\overline{\mathcal{O}}}(\mathcal{M}))^{S_g} = \mathcal{A}_g ,$$

is an isomorphism if and only if $n = 1$.

Proof. The proof is a computation of invariants in the exterior algebra of a free module over a *commutative* ring, which allows us to use standard methods of Commutative Algebra (all the results that we shall use can be found, for instance, in ref. [1]).

Let us start with the case $n = 1$.

(a) We can assume that $X = \operatorname{Spec}\mathcal{O}$, where $\mathcal{O}$ is a semilocal ring with g maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_g$, and then, that $\mathcal{N} = \mathcal{O} \cdot e$, $\mathcal{N}_i = \overline{\mathcal{O}} \cdot e_i$ (where

$$e_i = 1 \otimes \cdots \otimes \overset{\downarrow i)}{e} \otimes \cdots \otimes 1)$$

and $\mathcal{M} = \overline{\mathcal{O}} \cdot e_1 \oplus \cdots \oplus \overline{\mathcal{O}} \cdot e_g$.

Let us notice, first, that ϕ is an isomorphism if and only if it is an isomorphism when localized at every maximal ideal $\mathfrak{p}$ of $\mathcal{O}_g$. On the other hand, $\mathfrak{p}$ corresponds to a divisor $D = x_1 + \cdots + x_g$ and the fibre of $p\colon X^g \to S^g X$ over this point consists of the family $(x_1, \ldots, x_g)$ (some of the points can be equal) together with its permutations. It follows that we are reduced to considering only the localization of $\mathcal{O}$ at these particular points $(x_1, \ldots, x_g)$.

(b) We can assume that $\mathcal{O} = k[t]$ and $\mathcal{N} = k[t] \cdot e$. Since the completion morphism

$$(\mathcal{O}_g)_{\mathfrak{p}} \hookrightarrow (\widehat{\mathcal{O}_g})_{\mathfrak{p}}$$

is a faithfully flat morphism, we are reduced to showing that ϕ is an isomorphism after completing $\mathcal{O}_g$ at every maximal ideal.

Let $t \in \mathcal{O}$ be an element that takes different values $(\lambda_1, \ldots, \lambda_g)$ at the points $(x_1, \ldots, x_g)$ and such that $t - \lambda_i$ is a parameter at x_i (that is, it generates the maximal ideal of the local ring $\mathcal{O}_{\mathfrak{p}_i}$). Then, $k[t]$ is a subring of $\mathcal{O}$ and moreover, given two different maximal ideals $\mathfrak{p}_i \neq \mathfrak{p}_j$, the maximal ideals $\bar{\mathfrak{p}}_i = \mathfrak{p}_i \cap k[t]$ and $\bar{\mathfrak{p}}_j = \mathfrak{p}_j \cap k[t]$ of $k[t]$ are also different.

Let $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_g$, $\bar{I} = \bar{\mathfrak{p}}_1 \cap \cdots \cap \bar{\mathfrak{p}}_g$ be the intersection ideals and $\mathcal{O} \hookrightarrow \widehat{\mathcal{O}}$, $k[t] \hookrightarrow \widehat{k[t]}$ the faithfully flat morphisms of completion with respect to the ideals I, $\bar{I}$, respectively. Then we have

$$\widehat{k[t]} \simeq \prod_{i=1}^{g} \widehat{k[t]}_{\bar{\mathfrak{p}}_i} \simeq \prod_{i=1}^{g} \widehat{\mathcal{O}}_{\mathfrak{p}_i} \simeq \widehat{\mathcal{O}},$$

so that, if the theorem is true for $k[t]$ and the module $k[t] \cdot e$, it is also true for $\widehat{k[t]} = \widehat{\mathcal{O}}$ and $\widehat{k[t]} \cdot e = \widehat{\mathcal{O}} \cdot e$ and then it will be true for $\mathcal{O}$ and $\mathcal{N} = \mathcal{O} \cdot e$ by faithfull flatness.

(c) The case $\mathcal{O} = k[t]$ and $\mathcal{N} = k[t] \cdot e$. Now, for every $0 \leq p \leq g$ we have

$$\Lambda^p(\mathcal{M}) = \bigoplus_{i_1 < \cdots < i_p} \mathcal{N}_{i_1} \wedge \cdots \wedge \mathcal{N}_{i_p}$$

and S_g acts transitively by permutation of terms.

In fact, $\mathcal{N}_{i_1} \wedge \cdots \wedge \mathcal{N}_{i_p} = \sigma_{i_1 \ldots i_p}(\mathcal{N}_1 \wedge \cdots \wedge \mathcal{N}_p)$, $\sigma_{i_1 \ldots i_p} \in S_g$ being any permutation of the type

$$\sigma_{i_1 \ldots i_p} = \begin{pmatrix} 1 & \ldots & p & \ldots \\ i_1 & \ldots & i_p & \ldots \end{pmatrix}.$$

Then, an invariant element $m = \sum_{i_1 < \cdots < i_p} n_{i_1} \wedge \cdots \wedge n_{i_p}$ is characterized by $n_1 \wedge \cdots \wedge n_p$ and we have an isomorphism

$$(\mathcal{N}_1 \wedge \cdots \wedge \mathcal{N}_p)^{S_p \times S_{g-p}} \simeq (\Lambda^p(\mathcal{M}))^{S_g},$$
$$n_1 \wedge \cdots \wedge n_p \mapsto \sum_{\sigma \in S_g / (S_p \times S_{g-p})} \sigma(n_1 \wedge \cdots \wedge n_p),$$

where $S_p \times S_{g-p}$ denotes the subgroup of S_g consisting of the permutations leaving invariant the subset $\{1, \ldots, p\}$. In particular, $(\mathcal{N}_1)^{S_{g-1}} \simeq \mathcal{M}^{S_g}$.

Since $\mathcal{N}_1 \wedge \cdots \wedge \mathcal{N}_p = \overline{\mathcal{O}} \cdot e_1 \wedge \cdots \wedge e_p$, we have

$$(\mathcal{N}_1 \wedge \cdots \wedge \mathcal{N}_p)^{S_p \times S_{g-p}} \simeq \overline{\mathcal{O}}^{-S_p \times S_{g-p}},$$

where $\overline{\mathcal{O}}^{-S_p \times S_{g-p}}$ stands for the subset of such $f \in \overline{\mathcal{O}}$ that $(\sigma \times \mu)(f) = \operatorname{sign}(\sigma) \cdot f$ for every $(\sigma \times \mu) \in S_p \times S_{g-p}$. Taking $p = 1$, we obtain

$$\mathcal{M}^{S_g} \simeq \mathcal{N}_1^{S_{g-1}} \simeq \overline{\mathcal{O}}^{S_{g-1}},$$

and the original morphism

$$\phi \colon \Lambda^p_{\mathcal{O}_g}(\mathcal{M}^{S_g}) \to (\Lambda^p_{\overline{\mathcal{O}}}(\mathcal{M}))^{S_g} = \mathcal{A}_g$$

is now the morphism

$$\bar{\phi}_p \colon \wedge^p \bar{\mathcal{O}}^{S_{g-1}} \to \bar{\mathcal{O}}^{-S_p \times S_{g-p}}$$

described by

$$\bar{\phi}_p(f_1 \wedge \cdots \wedge f_p) = \sum_{\mu \in S_p} \operatorname{sign}(\mu) \sigma_{\mu(1)}(f_1) \ldots \sigma_{\mu(p)}(f_p),$$

where σ_i is the transposition of 1 and i.

Let us prove that $\bar{\phi}_p$ is an isomorphism: There is a commutative diagram

$$\begin{array}{ccc} \bar{\mathcal{O}}^{S_{g-1}} \otimes_{\mathcal{O}_g} \cdots \otimes_{\mathcal{O}_g} \bar{\mathcal{O}}^{S_{g-1}} & \xrightarrow{T} & \bar{\mathcal{O}}^{1 \times S_{g-p}} \\ H \big\downarrow & & \big\downarrow H' \\ \wedge^p_{\mathcal{O}_g} \bar{\mathcal{O}}^{S_{g-1}} & \xrightarrow{\bar{\phi}_p} & \bar{\mathcal{O}}^{-S_p \times S_{g-p}} \end{array},$$

where

$$\begin{aligned} T(f_1 \otimes \cdots \otimes f_p) &= \sigma_1(f_1) \ldots \sigma_p(f_p), \\ H(f_1 \otimes \cdots \otimes f_p) &= f_1 \wedge \cdots \wedge f_p, \\ H'(f) &= \sum_{\mu \in S_p} \operatorname{sign}(\mu)(\mu \times 1)(f). \end{aligned}$$

As $\mathcal{O} = k[t]$, $\bar{\mathcal{O}} = k[t_1, \ldots, t_g]$, if we denote

$$\begin{aligned} (s_1, \ldots, s_g) &= \text{symmetric functions of } (t_1, \ldots, t_g), \\ (\bar{s}_1, \ldots, \bar{s}_{g-1}) &= \text{symmetric functions of } (t_2, \ldots, t_g), \\ (s'_1, \ldots, s'_{g-p}) &= \text{symmetric functions of } (t_{p+1}, \ldots, t_g), \end{aligned}$$

we have

$$\begin{aligned} \mathcal{O}_g &= k[s_1, \ldots, s_g], \\ \bar{\mathcal{O}}^{S_{g-1}} &= k[t_1, \bar{s}_1, \ldots, \bar{s}_{g-1}] \xrightarrow{\sim} \mathcal{O}_g[t_1], \\ \bar{\mathcal{O}}^{1 \times S_{g-p}} &= k[t_1, \ldots, t_p, s'_1, \ldots, s'_{g-p}] \xrightarrow{\sim} \mathcal{O}_g[t_1, \ldots, t_p]. \end{aligned}$$

If follows that $\bar{\mathcal{O}}^{-S_p \times S_{g-p}}$ can be identified with the pth skew-symmetric tensors of the $\mathcal{O}_g$-module $\bar{\mathcal{O}}^{S_{g-1}} = \mathcal{O}_g[t_1]$ and the previous diagram reads

$$\begin{array}{ccc} \mathcal{O}_g[t_1] \otimes_{\mathcal{O}_g} \overset{p)}{\cdots} \otimes_{\mathcal{O}_g} \mathcal{O}_g[t_1] & \xrightarrow{\sim} & \mathcal{O}_g[t_1, \ldots, t_p] \\ H \big\downarrow & & \big\downarrow H' \\ \wedge^p_{\mathcal{O}_g} \mathcal{O}_g[t_1] & \xrightarrow[=]{\bar{\phi}_p} & \wedge^p_{\mathcal{O}_g} \mathcal{O}_g[t_1] \end{array},$$

where now H' is the skew-symmetrization operator, finishing the proof of the if part.

To complete the proof, we have to show that if $n > 1$, $\overline{\phi}_p$ is not an isomorphism. Let us write $\mathcal{N} = \bigoplus_{j=1}^{n} \mathcal{N}^j$ with $\mathcal{N}^j$ of rank 1, and

$$\mathcal{M}^j = \bigoplus_{i=1}^{g} (\mathcal{O} \otimes \cdots \otimes \overset{\downarrow i)}{\mathcal{N}^j} \otimes \cdots \otimes \mathcal{O})$$

so that $\mathcal{M} = \bigoplus_{j=1}^{n} \mathcal{M}^j$. Then

$$(\wedge_{\overline{\mathcal{O}}}(\mathcal{M}))^{S_g} = \bigoplus_{p_1+\cdots+p_s=p} (\wedge_{\overline{\mathcal{O}}}^{p_1}\mathcal{M}^1 \otimes \cdots \otimes \wedge_{\overline{\mathcal{O}}}^{p_s}\mathcal{M}^s)^{S_g},$$

$$\wedge_{\mathcal{O}_g}(\mathcal{M}^{S_g}) = \bigoplus_{p_1+\cdots+p_s=p} \wedge_{\mathcal{O}_g}^{p_1}(\mathcal{M}^1)^{S_g} \otimes \cdots \otimes \wedge_{\mathcal{O}_g}^{p_s}(\mathcal{M}^s)^{S_g}.$$

By the case $n = 1$, we have

$$\wedge_{\mathcal{O}_g}^{p_i}(\mathcal{M}^i)^{S_g} \simeq (\wedge_{\mathcal{O}_g}^{p_i}(\mathcal{M}^i))^{S_g}$$

and then

$$\wedge_{\mathcal{O}_g}(\mathcal{M}^{S_g}) = \bigoplus_{p_1+\cdots+p_s=p} (\wedge_{\mathcal{O}_g}^{p_1}\mathcal{M}^1)^{S_g} \otimes \cdots \otimes (\wedge_{\mathcal{O}_g}^{p_s}\mathcal{M}^s)^{S_g}.$$

But there are invariant elements in the tensor product $(\wedge_{\overline{\mathcal{O}}}^{p_1}\mathcal{M}^1 \otimes \cdots \otimes \wedge_{\overline{\mathcal{O}}}^{p_s}\mathcal{M}^s)^{S_g}$ which cannot be written as tensor products of invariant elements. This means that the morphism ϕ_p is not an isomorphism in this case. □

Corollary 1. *If (z, θ) are graded local coordinates on a supercurve $\mathcal{X}$ of dimension $(1,1)$, a system of graded local coordinates for $S^g\mathcal{X}$ is given by $(s_1, \ldots, s_g, \varsigma_1, \ldots, \varsigma_g)$, where $(s_1, \ldots, s_g)$ are the (even) symmetric functions of $(z_1, \ldots, z_g)$ and $(\varsigma_1, \ldots, \varsigma_g)$ are the odd symmetric functions defined by $\varsigma_h = \sum_{i=1}^{g} \sigma_i(\theta_1 \bar{s}_{h-1})$.*

3. Positive superdivisors

From this point, calligraphic types are reserved for graded ringed spaces and the structure ring sheaf of any ringed space will be denoted by $\mathcal{O}$ with the name of the ringed space as a subscript. For instance, $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ or simply $\mathcal{X}$ will mean a graded ringed space, whereas $(X, \mathcal{O}_X)$ or X will represent the underlying ordinary ringed space.

3.1. THE UNIVERSAL DIVISOR FOR AN ALGEBRAIC CURVE

This section is devoted to a summary of the theory of the variety of positive divisors and the universal divisor for a (ordinary) smooth proper algebraic curve X, and to show that the universal property still holds when the space of parameters is a superscheme. Suitable references are refs. [13] or [16].

In that case, positive divisors of degree g are unordered families of g points, and they are then parametrized by the space of such families, that is, by the symmetric product S^gX. This can be made precise through the notion of relative divisor.

If S is another scheme, positive relative divisors of $X \times S \to S$ of degree g are subschemes $Z \to X$ such that $\mathcal{O}_Z$ is a locally free $\mathcal{O}_S$-module of rank g. There is a nice positive relative divisor Z^{u} of degree g of $X \times S^gX \to S^gX$, whose fibre at a point $(x_1, \ldots, x_g) \in S^gX$ is the divisor $x_1 + \cdots + x_g$ of X defined by it. Z^{u} is called the *universal divisor* because the map

$$\mathrm{Hom}(S, S^gX) \to \mathrm{Div}_S^g(X \times S),$$
$$\phi \mapsto (1 \times \phi)^{-1}(Z^{\mathrm{u}}),$$

where $\mathrm{Div}_S^g(X \times S)$ denotes the set of positive relative divisors of degree g, is *one to one*. This means that each positive divisor can be obtained as a pull-back of the universal divisor; this statement is known as the *representability theorem* for the symmetric product.

But it turns out that the above theory is still true when a superscheme is allowed as the space of parameters, once the corresponding notion of positive relative divisor has been established.

Definition 4. Let X be an ordinary smooth curve and $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ a superscheme. A positive relative divisor of degree g of $X \times \mathcal{S} \to \mathcal{S}$ is a closed sub-superscheme $\mathcal{Z}$ of $X \times \mathcal{S}$ of codimension $(1,0)$ defined by a homogeneous ideal J of $\mathcal{O}_{X\times\mathcal{S}}$ such that $\mathcal{O}_{X\times\mathcal{S}}/J$ is a locally free $\mathcal{O}_{\mathcal{S}}$-module of rank $(g,0)$.

The ideal J of a positive relative divisor of degree g is then locally generated by an element of the type

$$f = z^g - a_1 z^{g-1} + \cdots + (-1)^g a_g, \tag{2}$$

where the a_i's are even elements in $\mathcal{O}_{\mathcal{S}}$, and $\mathcal{O}_{X\times\mathcal{S}}/J$ is a free $\mathcal{O}_{\mathcal{S}}$-module with basis $(1, z, \ldots, z^{g-1})$.

The representability theorem now reads

Theorem 2. *Let X be a smooth proper curve over a field k and Z^{u} the universal divisor. The map*

$$\mathrm{Hom}(\mathcal{S}, S^gX) \to \mathrm{Div}_{\mathcal{S}}^g(X \times \mathcal{S}),$$
$$\phi \mapsto (1 \times \phi)^{-1}(Z^{\mathrm{u}}), \tag{3}$$

where $\mathrm{Div}_{\mathcal{S}}^g(X \times \mathcal{S})$ *denotes the set of positive relative divisors of degree g, is one to one for every superscheme $\mathcal{S}$.* □

The proof of the representability theorem for ordinary schemes applies to this case with only minor changes.

There are two key points for the proof of this theorem. The first one is the construction of the universal divisor, which can be done as follows: If $\pi_i\colon X^g \to X$ is the ith projection and Δ_i is the positive relative divisor of $X \times X^g \to X^g$ obtained by pull-back of the diagonal $\Delta \subset X \times X$ throughout $1 \times \pi_i\colon X \times X^g \to X \times X$ we can prove there is a unique positive relative divisor Z^{u} of $X \times S^g X \to S^g X$ such that

$$(1 \times p)^{-1} Z^{\mathrm{u}} = \Delta_1 + \cdots + \Delta_g ,$$

where $p\colon X^g \to S^g X$ is the natural projection. This divisor Z^{u} is the universal divisor.

The second key point is the so-called "determinant morphism" $\mathcal{S} \to S^g \mathcal{Z}$, $\mathcal{Z}$ being a positive relative divisor of degree g because its composition with $S^g \mathcal{Z} \to S^g X$ provides the inverse mapping of (3) (see ref. [16]). The determinant morphism for the locally free $\mathcal{O}_{\mathcal{S}}$-module of rank $(g, 0)$ $\mathcal{O}_{\mathcal{Z}}$ is defined as follows: Each element b in the invariant sheaf $(\mathcal{O}_{\mathcal{Z}})_g = (\mathcal{O}_{\mathcal{Z}}^{\otimes g})^{S_g}$ acts on the $\mathcal{O}_{\mathcal{S}}$-module $\wedge_{\mathcal{O}_{\mathcal{S}}} \mathcal{O}_{\mathcal{Z}}$ of rank $(1, 0)$ as the multiplication by a well-determined element $\det(b)$ in $\mathcal{O}_{\mathcal{S}}$. This gives rise to a morphism of sheaves $(\mathcal{O}_{\mathcal{Z}})_g \to \mathcal{O}_{\mathcal{S}}$, and to a morphism of schemes $\mathcal{S} \to S^g \mathcal{Z}$. The determinant morphism provides the inverse mapping of (3) because, if b is an even element in $\mathcal{O}_{\mathcal{Z}}$,

$$b_i = 1 \otimes \cdots \otimes \overset{\downarrow i)}{b} \otimes \cdots \otimes 1 \in \mathcal{O}_{\mathcal{Z}}^{\otimes g},$$

and we denote by $s_i(b)$ the symmetric functions of $b_1, \ldots, b_g$, we have that

$$a_i = \det(s_i(b)) \quad (i = 1, \ldots, g),$$

where $z^g - a_1 z^{g-1} + \cdots + (-1)^g a_g$ is the characteristic polynomial of b acting on $\mathcal{O}_{\mathcal{Z}}$ by multiplication [compare with (2)].

3.2. POSITIVE SUPERDIVISORS ON SUPERCURVES

The above discussion is based on a trivial but important point: positive divisors are families of points. Even in the relative case, positive relative divisors of degree 1 are "S-points", that is, sections of $X \times S \to S$, and for this reason, positive divisors of degree g are parametrized by the symmetric product $S^g X$ and the universal divisor.

For a supercurve $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$, a similar notion could be established, by defining positive relative superdivisors of $\mathcal{X} \times \mathcal{S} \to \mathcal{S}$ ($\mathcal{S}$ being an arbitrary superscheme), as closed sub-superschemes of $\mathcal{X} \times \mathcal{S}$ of codimension $(1, 0)$ flat over the base superscheme.

This definition has two drawbacks. The first one is that "$\mathcal{S}$-points" are not superdivisors in that sense because they have codimension $(1, 1)$ and not codimension $(1, 0)$ as superdivisors do [23]. The second one is that we cannot ensure that they are pull-backs of a suitable universal superdivisor.

We have thus modified the notion of positive relative superdivisors in order to fulfill the second requirement as follows:

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a smooth supercurve, and $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ a superscheme.

Definition 5. A positive relative superdivisor of degree g of $\mathcal{X} \times \mathcal{S} \to \mathcal{S}$ is a closed sub-superscheme $\mathcal{Z}$ of $\mathcal{X} \times \mathcal{S}$ of codimension $(1,0)$ whose reduction $\hat{\mathcal{Z}} = \mathcal{Z} \times_{\mathcal{X}} X$ is a positive relative divisor of degree g of $X \times \mathcal{S} \to \mathcal{S}$ (see definition 4).

Even with our definition, "$\mathcal{S}$-points" are not superdivisors, but as we shall see later, there is a close relationship between them, at least for SUSY-curves.

Positive relative superdivisors can be described locally in a rather precise way in the case of a smooth supercurve of dimension $(1,1)$.

In this case, the natural morphism $\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_X$ induces an isomorphism $(\mathcal{O}_{\mathcal{X}})_0 \simeq \mathcal{O}_X$, so that $\mathcal{O}_{\mathcal{X}}$ is a module over $\mathcal{O}_X$, there exists a canonical projection $\mathcal{X} \to X$ and $\mathcal{O}_{\mathcal{X}}$ is in a natural way an exterior algebra $\mathcal{O}_{\mathcal{X}} = \bigwedge_{\mathcal{O}_X} \mathcal{L}$, where $\mathcal{L} = (\mathcal{O}_{\mathcal{X}})_1$ is a line bundle over the ordinary curve X.

Lemma 2. *Let $\mathcal{X}$ be a smooth supercurve of dimension $(1,1)$. A closed subsuperscheme $\mathcal{Z}$ of $\mathcal{X} \times \mathcal{S}$ of codimension $(1,0)$ defined by a homogeneous ideal J of $\mathcal{O}_{\mathcal{X} \times \mathcal{S}}$ is a positive relative superdivisor of degree g if and only if the following conditions hold:*

(i) $\mathcal{O}_{\mathcal{Z}} = \mathcal{O}_{\mathcal{X} \times \mathcal{S}}/J$ is a locally free $\mathcal{O}_{\mathcal{S}}$-module of dimension (g,g).

(ii) If (z,θ) is a system of graded local coordinates, J can be locally generated by an element of type

$$f = z^g - (a_1 + \theta b_1) z^{g-1} + \cdots + (-1)^g (a_g + \theta b_g),$$

where the a_i's are even and the b_j's are odd elements in $\mathcal{O}_{\mathcal{S}}$.

Proof. Let $\mathcal{Z}$ be a positive relative superdivisor of degree g defined by a homogeneous ideal J of $\mathcal{O}_{\mathcal{X} \times \mathcal{S}}$ and let us consider a system of relative local coordinates (z,θ). Then, the reduction $\hat{\mathcal{Z}} = \mathcal{Z} \times_{\mathcal{X}} X$ is a positive relative divisor of degree g of $X \times \mathcal{S} \to \mathcal{S}$ defined by the image $\hat{J}$ of J by the morphism $\pi\colon \mathcal{O}_{\mathcal{X} \times \mathcal{S}} \to \mathcal{O}_{X \times \mathcal{S}}$, so that an element $f \in J$ generates J if and only if $\hat{J}$ is generated by $\hat{f} = \pi(f)$. Since $\hat{J}$ defines a positive relative divisor of degree g of $X \times \mathcal{S} \to \mathcal{S}$, then $\hat{J}$ has a generator of type $\hat{f} = z^g - a_1 z^{g-1} + \cdots + (-1)^g a_g$, where the a_i's are even elements in $\mathcal{O}_{\mathcal{S}}$ [see eq. (2)], and $\mathcal{O}_{\hat{\mathcal{Z}}} = \mathcal{O}_{X \times \mathcal{S}}/\hat{J}$ is a free $\mathcal{O}_{\mathcal{S}}$-module with basis $(1, z, \ldots, z^g)$. This means that $\mathcal{O}_{\hat{\mathcal{Z}}} \simeq \mathcal{O}_{\mathcal{S}}[z]/(\hat{f})$. It follows that there is a generator of J of the form $f = \hat{f} + \theta \cdot d$ and that $d \equiv q(z) \pmod{\hat{J}}$ for a certain polynomial $q(z)$ of degree less than g. In consequence, the element $\hat{f} + \theta q(z)$ generates J and is of the predicted type. An easy computation now shows that $\mathcal{O}_{\mathcal{Z}}$ is a rank (g,g) free $\mathcal{O}_{\mathcal{S}}$-module with basis $(1, z, \ldots, z^{g-1}, \theta, \theta z, \ldots, \theta z^{g-1})$.

The converse is straightforward. □

3.3. THE FUNCTOR OF POSITIVE SUPERDIVISORS ON A SUPERCURVE

Let $(\mathcal{X}, \mathcal{Z})$ be a supercurve. For every superscheme $\mathcal{S}$ let us denote by $\mathrm{Div}^g_{\mathcal{S}}(\mathcal{X} \times \mathcal{S})$ the set of positive relative superdivisors of degree g of $\mathcal{X} \times \mathcal{S} \to \mathcal{S}$. If $\varphi: \mathcal{S}' \to \mathcal{S}$ is a morphism of superschemes, and $\mathcal{Z}$ is a positive relative divisor of degree g of $\mathcal{X} \times \mathcal{S} \to \mathcal{S}$, $(1 \times \varphi)^{-1}\mathcal{Z}$ is a positive relative divisor of degree g of $\mathcal{X} \times \mathcal{S}' \to \mathcal{S}'$. In categorial language this essentially means that

$$\mathcal{S} \to \mathrm{Div}^g_{\mathcal{S}}(\mathcal{X} \times \mathcal{S})$$

is a functor.

We whish to show that when $\mathcal{X}$ has dimension $(1, 1)$, the above functor is *representable* in a similar sense to that of theorem 2. A proof is given in the next section.

4. The representability theorem for positive superdivisors on a supercurve of dimension (1, 1)

In what follows, we consider only supercurves $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ which are smooth, proper and of dimension $(1, 1)$. This last condition means that the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is canonically isomorphic with $\mathcal{O}_X \oplus \mathcal{L}$ for a certain line bundle $\mathcal{L}$ on the ordinary underlying curve X.

4.1. THE SUPERCURVE OF POSITIVE SUPERDIVISORS OF DEGREE 1

Let $\mathcal{S} = (\mathrm{Spec}\, B, \mathcal{B})$ be an affine superscheme and $\mathcal{Z} = (Z, \mathcal{O}_{\mathcal{Z}}) \hookrightarrow \mathcal{X} \times \mathcal{S} \to \mathcal{S}$ a relative superdivisor of degree 1. The structure sheaf $\mathcal{O}_{\mathcal{Z}}$ is a quotient of the structure sheaf $(\mathcal{O}_X \oplus \mathcal{L}) \otimes_k \mathcal{B}$ of $\mathcal{X} \times \mathrm{Spec}\, B$. We also have that $\mathcal{O}_{\mathcal{Z}} \simeq \mathcal{B} \oplus \overline{\mathcal{L}}$, where $\overline{\mathcal{L}}$ is the image of $\mathcal{L} \otimes_k \mathcal{B}$ in $\mathcal{O}_{\mathcal{Z}}$, since $\mathcal{O}_{\hat{\mathcal{Z}}} \simeq \mathcal{B}$, because $\mathcal{Z}$ is a superdivisor of degree 1. Moreover, $\overline{\mathcal{L}} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{B}$, where $\mathcal{B}$ is an $\mathcal{O}_X$-algebra trough the natural morphism $f: \mathcal{O}_X \to \mathcal{O}_{\hat{\mathcal{Z}}} \simeq \mathcal{B}$, so that it is a locally free rank-1 $\mathcal{B}$-module.

It is now clear that the superdivisor $\mathcal{Z}$ is characterized by the morphism $f: \mathcal{O}_X \to \mathcal{B}$ together with a morphism $\tilde{f}: \mathcal{O}_X \to \mathcal{B} \oplus \overline{\mathcal{L}}$ extending f. That is, $\mathcal{Z}$ is defined by a morphism $f: \mathcal{O}_X \to \mathcal{B}$ and a derivation $\Delta: \mathcal{O}_X \to \overline{\mathcal{L}}_0 = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{B}_1$. But Δ can be understood as an element $f_\Delta \in \mathrm{Hom}_{\mathcal{O}_X}(\kappa_X, \overline{\mathcal{L}}_0) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\kappa_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}, \mathcal{B}_1)$ (where κ_X is the canonical sheaf of X), so that the couple (f, Δ) is equivalent to a graded ring morphism $g: \mathcal{O}_X \oplus (\kappa_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \to \mathcal{B}$.

The above discusion remains true for arbitrary (non-affine) superschemes $\mathcal{S}$. This means that the supercurve $\mathcal{X}^c = \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{L}^c)$, where $\mathcal{L}^c = \kappa_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}$, will represent the functor of superdivisors of degree 1 of the supercurve $\mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{L})$. The universal divisor, $\mathcal{Z}^u_1 \hookrightarrow \mathcal{X} \times \mathcal{X}^c$, will be the divisor corresponding to the identity morphism $\mathrm{Id}: \mathcal{S} = \mathcal{X}^c \to \mathcal{X}^c$. One can compute this superdivisor as above and obtain that it is the closed subsuperscheme whose ideal sheaf is the kernel of the graded ring morphism:

$$\overline{\partial}\colon (\mathcal{O}_X \oplus \mathcal{L}) \otimes_k (\mathcal{O}_X \oplus \mathcal{L}^c) = \wedge_{\mathcal{O}_X \otimes_k \mathcal{O}_X} [(\mathcal{L} \otimes_k \mathcal{O}_X) \oplus (\mathcal{O}_X \otimes_k \mathcal{L}^c)] \\ \to \wedge_{\mathcal{O}_X} (\mathcal{L} \oplus \mathcal{L}^c)$$

given by $a \otimes b \mapsto a \cdot b \oplus b \cdot d(a)$ on $\mathcal{O}_X \otimes_k \mathcal{O}_X$ (taking into account that $b \cdot d(a)$ is a local section of $\kappa_X \simeq \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^c$) and as the natural morphisms on the remaining components. Moreover, if $U \subset X$ is an affine open subset and $z \in \mathcal{O}_X(U)$ is a local parameter, and if $\mathcal{L}$ is trivial on U, $\mathcal{L}_{|U} \simeq \theta \cdot \mathcal{O}_{X|U}$, then $\mathcal{L}^c$ is trivial on U generated by $\theta^c = \omega_\theta \cdot \mathrm{d}z$, $\omega_\theta \in \Gamma(U, \mathcal{L}^{-1})$ being the dual basis of θ. If $\mathcal{U} = \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{L}) \subset \mathcal{X}$ and $\mathcal{U}^c = \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{L}^c) \subset \mathcal{X}^c$, the restriction of the universal superdivisor $\mathcal{Z}_1^u$ to $\mathcal{U} \times \mathcal{U}^c$ is given by the local equation:

$$z_1 - z_2 - \theta \otimes \theta^c = 0, \tag{4}$$

where $z_1 = z \otimes 1$ and $z_2 = 1 \otimes z$.

The above discussion can be summarized as follows: Let $\mathcal{X} = (X, \mathcal{O}_\mathcal{X} = \mathcal{O}_X \oplus \mathcal{L})$ be a smooth proper supercurve of dimension $(1,1)$.

Definition 6. The supercurve of positive divisors of degree 1 on $\mathcal{X}$ is the supercurve of dimension $(1,1)$ defined as $\mathcal{X}^c = (X, \mathcal{O}_X \oplus \mathcal{L}^c)$ where $\mathcal{L}^c = \kappa_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}$. This supercurve is also called the supercurve of conjugate fermions on $\mathcal{X}$.

Definition 7. The universal positive superdivisor of degree 1 is the relative superdivisor $\mathcal{Z}_1^u$ of $\mathcal{X} \times \mathcal{X}^c \to \mathcal{X}^c$ defined by the ideal sheaf $\mathrm{Ker}\, \overline{\partial}$ earlier considered. If (z, θ) are graded local coordinates for $\mathcal{X}$, the corresponding local equation of $\mathcal{Z}_1^u$ is $z_1 - z_2 - \theta \otimes \theta^c = 0$, where $z_1 = z \otimes 1$, $z_2 = 1 \otimes z$ and $\theta^c = \omega_\theta \cdot \mathrm{d}z$.

Theorem 3. *The morphism of functors:*

$$\boldsymbol{\Theta}\colon \mathrm{Hom}(\mathcal{S}, \mathcal{X}^c) \to \mathrm{Div}^1_\mathcal{S}(\mathcal{X} \times \mathcal{S}), \\ \varphi \mapsto (1 \times \varphi)^{-1}(\mathcal{Z}_1^u),$$

is a functorial isomorphism.

By this representability theorem, the supercurve $\mathcal{X}^c$ of conjugate fermions parametrizes positive superdivisors of degree 1 on the original supercurve $\mathcal{X}$. That means that positive superdivisors of degree 1 on $\mathcal{X}$ are not points of $\mathcal{X}$ as happens in the ordinary case, but rather points of another supercurve $\mathcal{X}^c$ with the same underlying ordinary curve X.

4.2. POSITIVE SUPERDIVISORS OF DEGREE 1 ON A SUSY-CURVE

This section will explore the relationship between points and positive superdivisors of degree 1 for a SUSY-curve (supersymmetric curve). This relationship was first described by Manin (see ref. [23]), but it can be enlightened by means of the supercurve of positive superdivisors of degree 1 defined above. Let us start by recalling some definitions and elementary properties of SUSY-curves. More details can be found in Manin [21–24], Batchelor and Bryant [3], Falqui and Reina [9], Giddings and Nelson [11,12], Bartocci, Bruzzo and Hernández Ruipérez [2], Bruzzo and Domínguez Pérez [6], or LeBrun, Rothstein, Yat-Sun Poon and Wells [18,19].

Let $\mathcal{S} = (S, \mathcal{O}_S)$ be a superscheme.

Definition 8. A supersymmetric curve or SUSY-curve over $\mathcal{S}$ is a proper smooth morphism $\mathcal{X} = (X, \mathcal{O}_\mathcal{X}) \to \mathcal{S}$ of superschemes of relative dimension $(1,1)$ endowed with a locally free submodule $\mathcal{D}$ of rank $(0,1)$ of the relative tangent sheaf $T_{\mathcal{X}/\mathcal{S}} = \mathcal{D}er_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{X})$ such that the composition map

$$\mathcal{D} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{D} \xrightarrow{[\,,\,]} \mathcal{D}er_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{X}) \to \mathcal{D}er_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{X})/\mathcal{D}$$

is an isomorphism of $\mathcal{O}_\mathcal{X}$-modules (see, for instance, ref. [19]).

If $\mathcal{X} = (X, \mathcal{O}_\mathcal{X}, \mathcal{D})$ is a SUSY-curve, X can be covered by affine open subsets $U \subseteq X$ with local relative coordinates (z, θ) such that $\mathcal{D}$ is locally generated by $D = \partial/\partial\theta + \theta\partial/\partial z$. These coordinates are called *conformal.*

There is a natural isomorphism $\mathcal{D}^* \xrightarrow{\sim} \mathrm{Ber}_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{X})$ and a "Berezinian differential"

$$\partial : \Omega^1_{\mathcal{X}/\mathcal{S}} \to \mathcal{D}^* \simeq \mathrm{Ber}_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{X}),$$

which is nothing but the natural projection induced by the immersion $\mathcal{D} \to T_{\mathcal{X}/\mathcal{S}}$. In conformal coordinates ∂ is described by $\partial(\mathrm{d}f) = [\mathrm{d}z \otimes \partial/\partial\theta] \cdot D(f)$, where $[\mathrm{d}z \otimes \partial/\partial\theta]$ denotes the local basis of $\mathrm{Ber}_{\mathcal{O}_\mathcal{S}}\, \mathcal{O}_\mathcal{X}$ determined by (z, θ) (see refs. [15,23]).

If $(X, \mathcal{O}_\mathcal{X}, \mathcal{D})$ is a (single) SUSY-curve, that is, a SUSY-curve over a point, we have that $\mathcal{O}_\mathcal{X} = \bigwedge_{\mathcal{O}_X}(\mathcal{L})$, and there are isomorphisms $\mathcal{D} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_X \xrightarrow{\sim} \mathcal{L}^{-1}$ and

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\sim} \kappa_X .$$

This isomorphism if often called a spin structure on X. Conversely, a spin structure induces a conformal structure, so that a conformal structure on a proper smooth supercurve is equivalent to a spin structure on it.

Now, there is a geometrical characterization of SUSY-curves in terms of superdivisors:

Theorem 4. *Let $\mathcal{X}$ be a supercurve of dimension* $(1,1)$. *Then $\mathcal{X}$ is a SUSY-curve if and only if there is an isomorphism of supercurves $\mathcal{X} \simeq \mathcal{X}^c$ between $\mathcal{X}$ and the supercurve of positive superdivisors of degree 1 (conjugate fermions) on it inducing the identity on X. Moreover, there is a one-to-one correspondence between such isomorphisms and spin structures on X.*

Proof. If $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \oplus \mathcal{L}$, then the structure sheaf of $\mathcal{X}^c$ is $\mathcal{O}_X \oplus (\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \kappa_X)$, so that an isomorphism $\mathcal{X} \simeq \mathcal{X}^c$ inducing the identity on X is nothing but a $\mathcal{O}_X$-module isomorphism $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \kappa_X \simeq \mathcal{L}$, that is, an isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \kappa_X$. □

Theorem 3 and the above result mean that for SUSY-curves, $\mathcal{S}$-points are equivalent to relative positive superdivisors of degree 1 on $\mathcal{X} \times \mathcal{S} \to \mathcal{S}$, as Manin claimed in ref. [23], and the universal relative positive superdivisor of degree 1 gives in this case nothing but Manin's superdiagonal:

Let $\mathcal{X}$ be a SUSY-curve. If Δ denotes the ideal of the diagonal immersion $\Delta\colon \mathcal{X} \hookrightarrow \mathcal{X} \times \mathcal{X}$, the kernel of the composition

$$\Delta \to \Delta/\Delta^2 \simeq \Delta_* \Omega^1_{\mathcal{X}} \xrightarrow{\partial} \Delta_* \operatorname{Ber}(\mathcal{O}_{\mathcal{X}})$$

is a homogeneous ideal $\mathcal{I}$ of $\mathcal{O}_{\mathcal{X} \times \mathcal{X}}$ thus defining a subsuperscheme Δ^s called the *superdiagonal*.

Lemma 3 (Manin [23]). *The superdiagonal $\Delta^s = (X, \mathcal{O}_{\mathcal{X}\times\mathcal{X}}/\mathcal{I})$ is a closed subsuperscheme of codimension* $(1,0)$. *In conformal coordinates* (z,θ), *it can be described by the equation*

$$z_1 - z_2 - \theta_1\theta_2 = 0,$$

where as usual $z_1 = 1 \otimes z$ and $z_2 = z \otimes 1$. □

According to lemma 2, the superdiagonal is a positive superdivisor. A simple local computation shows that actually we have:

Theorem 5. *Let $\mathcal{X}$ be a SUSY-curve, $\psi\colon \mathcal{X} \simeq \mathcal{X}^c$ the natural isomorphism between X and the supercurve of positive superdivisors of degree 1 (conjugate fermions), and $1 \times \psi\colon \mathcal{X} \times \mathcal{X} \simeq \mathcal{X} \times \mathcal{X}^c$ the induced isomorphism. Then*

$$\Delta^s = (1 \times \psi)^{-1}(\mathcal{Z}_1),$$

that is, the isomorphism $\psi\colon \mathcal{X} \simeq \mathcal{X}^c$ given by the spin structure transforms by inverse image the universal positive superdivisor of degree 1 into Manin's superdiagonal. □

4.3. THE SUPERSCHEME OF POSITIVE SUPERDIVISORS OF DEGREE g

Let $\mathcal{X}$ be a smooth proper supercurve of dimension $(1,1)$ as above.

Definition 9. The superscheme of positive superdivisors of degree g of $\mathcal{X}$ is the supersymmetric product $S^g\mathcal{X}^c$ of the supercurve $\mathcal{X}^c$ of positive superdivisors of degree 1.

The universal superdivisor $\mathcal{Z}_g^u$ of $\mathcal{X} \times S^g\mathcal{X}^c$ is constructed as follows: let us consider the natural projections

$$\pi_i\colon \mathcal{X} \times \mathcal{X}^c \times \overset{g}{\cdots} \times \mathcal{X}^c \to \mathcal{X} \times \mathcal{X}^c, \qquad (x, x_1^c, \ldots, x_g^c) \mapsto (x, x_i^c),$$

the positive superdivisors of degree 1, $\mathcal{Z}_i = \pi_i(\mathcal{Z}_1^u) \subset \mathcal{X} \times (\prod_{i=1}^g \mathcal{X}^c)$ and the positive superdivisor of degree g, $\mathcal{Z} = \mathcal{Z}_1 + \cdots + \mathcal{Z}_g$.

Lemma 4. *There exists a unique positive relative superdivisor $\mathcal{Z}_g^u$ of degree g of $\mathcal{X} \times S^g\mathcal{X}^c \to S^g\mathcal{X}^c$, such that $\pi^*(\mathcal{Z}_g^u) = \mathcal{Z}$, where π is the natural morphism*

$$\pi\colon \mathcal{X} \times \left(\prod_{i=1}^{g} \mathcal{X}^c\right) \to \mathcal{X} \times S^g\mathcal{X}^c.$$

Proof. One has only to prove that $\mathcal{Z}_g^u = \pi(\mathcal{Z})$ is the desired superdivisor. This can be done locally, so that we can assume that $\mathcal{X} = \operatorname{Spec} A$ is affine and the line bundles $\mathcal{L}$ and κ_X are trivially generated by θ and $\mathrm{d}z$, respectively. Then, the local equation of $\mathcal{Z}_1^u$ is $z \otimes 1 - 1 \otimes z - \theta \otimes \theta^c = 0$ [see eq. (4)], and $\mathcal{Z}$ is the superdivisor defined by the equation

$$0 = \prod_{i=1}^{g} (z - z_i - \theta\theta_i^c) = z^g - (s_1 + \theta \cdot \varsigma_1) z^{g-1} + \cdots + (-1)^g (s_g + \theta \cdot \varsigma_g),$$

where $z_i = \pi_i^*(1 \otimes z), \theta_i^c = \pi_i^*(1 \otimes \theta^c)$ and s_i, ς_i are the even and odd symmetric functions corresponding to z and θ^c (see corollary 1). It follows that this last equation is also the local equation of $\mathcal{Z}_g^u$ in $\mathcal{X} \times S^g\mathcal{X}^c$ and one can readily check that $\pi^*(\mathcal{Z}_g^u) = \mathcal{Z}$. □

4.4. THE REPRESENTABILITY THEOREM

This subsection will justify the above definitions by displaying the representability theorem:

Theorem 6. *The pair* $(S^g\mathcal{X}^c, \mathcal{Z}^u_g)$ *represents the functor of relative positive superdivisors of degree* g *of* $\mathcal{X}$, *that is, the natural map:*

$$\phi\colon \operatorname{Hom}(\mathcal{S}, S^g\mathcal{X}^c) \to \operatorname{Div}^g_{\mathcal{S}}(\mathcal{X}\times\mathcal{S}),$$
$$f \mapsto (1\times f)^*\mathcal{Z}^u_g,$$

is a functorial isomorphism for every superscheme $\mathcal{S}$.

Proof.

(1) ϕ *is injective:* Let $U = \operatorname{Spec} A \subset X$ be an open subscheme of the underlying ordinary curve X, such that κ_X and $\mathcal{L}$ are trivial generated by $\mathrm{d}z$ and θ, respectively. Let us consider the affine open subsuperschemes $\mathcal{U} = \operatorname{Spec}(A\oplus\theta\cdot A) \hookrightarrow \mathcal{X}$ and $\mathcal{U}^c = \operatorname{Spec}(A\oplus\theta^c\cdot A) \hookrightarrow \mathcal{X}^c$, where $\theta^c = \mathrm{d}z\otimes\omega_\theta \in \Gamma(U, \kappa_X\otimes\mathcal{L}^{-1}) = \Gamma(U, \mathcal{L}^c)$.

Now, $S^g\mathcal{U}^c \hookrightarrow S^g\mathcal{X}^c$ is an affine open subsuperscheme and the symmetric functions $s_i(z)$, $\varsigma_i(z,\theta^c)$ $(i = 1,\dots,g)$ form a graded system of parameters for the graded ring $S^g_k(A\oplus\theta^c\cdot A)$. Let us denote it simply by s_i, ς_i.

The family of the affine open subsuperschemes $S^g\mathcal{U}^c$ so obtained (when U ranges over the affine open subschemes of X where κ_X and $\mathcal{L}$ are trivial) is an open covering of $S^g\mathcal{X}^c$ by affine open subsuperschemes such that the universal positive superdivisor of $\mathcal{U}\times S^g\mathcal{U}^c \to \mathcal{U}$ is the closed subsuperscheme $\mathcal{Z}^u_{\mathcal{U}}$ defined by the equation

$$z^g - (s_1 + \theta\cdot\varsigma_1)z^{g-1} + \cdots + (-1)^g(s_g + \theta\cdot\varsigma_g) = 0.$$

Then one has that for these affine open subsuperschemes the map

$$\phi_{\mathcal{U}}\colon \operatorname{Hom}(\mathcal{S}, S^g\mathcal{U}^c) \to \operatorname{Div}^g_{\mathcal{S}}(\mathcal{U}\times\mathcal{S}),$$
$$f \mapsto (1\times f)^*\mathcal{Z}^u_{\mathcal{U}},$$

is injective: In fact, we can assume that $\mathcal{S}$ is affine, $\mathcal{S} = \operatorname{Spec} B$. Now, the morphisms $f\colon \mathcal{S} \to S^g\mathcal{U}^c$ are determined by the inverse images of the symmetric functions s_i, ς_i. But these inverse images are determined by $(1\times f)^*\mathcal{Z}^u_{\mathcal{U}}$ since the coefficients of the characteristic polynomial of $z\otimes 1$ acting by multiplication on the $B[\theta]$-module $\mathcal{O}_{(1\times f)^*\mathcal{Z}^u_{\mathcal{U}}}$ are $(-1)^i(f^*(s_i) + \theta\cdot f^*(\varsigma_i))$. This allows us to conclude.

A straightforward consequence of this fact is that the map ϕ of the statement is injective for every superscheme $\mathcal{S}$.

(2) ϕ *is an epimorphism:* It is sufficient to prove that, given a relative positive superdivisor of degree g, $\mathcal{Z}\subset\mathcal{X}\times\mathcal{S}\to\mathcal{S}$, for every geometric point $p\in\mathcal{S}$ there exist an open neighbourhood, $\mathcal{V}\subset\mathcal{S}$, and a morphism $f_{\mathcal{V}}\colon \mathcal{V}\to S^g\mathcal{X}^c$ such that $(1\times f_{\mathcal{V}})^*(\mathcal{Z}^u_g) = \mathcal{Z}\cap(\mathcal{X}\times\mathcal{V}) = \mathcal{Z}_{\mathcal{V}}$, for, in that case, these morphisms define a morphism $f\colon \mathcal{S}\to S^g\mathcal{X}^c$ fulfilling $(1\times f)^*(\mathcal{Z}^u_g) = \mathcal{Z}$ by virtue of the preceding subsection. Let $\pi\colon \mathcal{X}\times\mathcal{S}\to\mathcal{S}$ be the natural projection and $U = \operatorname{Spec} A \subset X$ an affine subscheme where κ_X and $\mathcal{L}$ are trivial and such that (with the notation of

the beginning of this section) the affine open subsuperscheme $\mathcal{U} \subset \mathcal{X}$ contains the superdivisor $\pi^{-1}(p) \cap \mathcal{Z} \hookrightarrow \mathcal{X}$. Then, $\mathcal{W} = \mathcal{S} - \pi(\mathcal{Z} - \mathcal{Z} \cap (\mathcal{U} \times \mathcal{S}))$ is open, because π is a proper morphism, and it contains the point $p \in \mathcal{S}$. Let $\mathcal{V} = \operatorname{Spec} B \subset \mathcal{S}$ be an affine open subsuperscheme containing p and contained in $\mathcal{W}$. By construction, if we put $\mathcal{Z}_{\mathcal{V}} = \mathcal{Z} \cap \pi^{-1}(\mathcal{V})$, then $\mathcal{Z}_{\mathcal{V}}$ is a relative positive divisor of degree g of $\mathcal{U} \times \mathcal{V} \to \mathcal{V}$, so that it is affine, $\mathcal{Z}_{\mathcal{V}} = \operatorname{Spec} C$. Let $\mathrm{d}z$, θ be generators of κ_X and $\mathcal{L}$, as usual. Now, according to the definition of superdivisor, the ring C of $\mathcal{Z}_{\mathcal{V}}$ is a locally free module over $B[\theta]$ of rank g and $\overline{C} = C/\theta \cdot C$ is the ring of an ordinary divisor of degree g of $U \subset X$.

Let us consider the morphism $f_{\mathcal{V}}\colon \operatorname{Spec} B = \mathcal{V} \to S^g\mathcal{U}^{\mathrm{c}} = \operatorname{Spec} S^g_k(A \oplus \theta^{\mathrm{c}} \cdot A)$ induced by the ring morphism $f^*_{\mathcal{V}}\colon S^g_k(A \oplus \theta^{\mathrm{c}} \cdot A) \to B$ defined, by means of the determinant morphism, as follows: Let $S^g_k A \to B$ be the determinant morphism defined by the quotient ring $\overline{C}$ of $A \otimes_k B$. This morphism endows B with a structure of $S^g_k A$-algebra. But, by lemma 1, one has $S^g_k(A \oplus \theta^{\mathrm{c}} \cdot A) = \wedge_{S^g_k A} M$ for a certain free $S^g_k A$-module M generated by the odd symmetric functions ς_i; then, by the universal property of the exterior algebra, defining $f^*_{\mathcal{V}}$ is equivalent to giving a homogeneous morphism of degree zero of $S^g_k A$-modules, $M \to B$. This morphism is actually characterized by the images of the functions ς_i ($i = 1, \dots, g$), and we define these images as the odd coefficients of the characteristic polynomial of $z \otimes 1$ acting on the $B[\theta]$-module C by multiplication; this means that, if the characteristic polynomial is $z^g - (a_1 + \theta \cdot b_1) z^{g-1} + \cdots + (-1)^g (a_g + \theta \cdot b_g)$, then we define $f^*_{\mathcal{V}}(\varsigma_i) = b_i$.

Moreover, one also has that $f^*_{\mathcal{V}}(s_i) = a_i$ and then $(1 \times f)^*_{\mathcal{V}}(\mathcal{Z}^{\mathrm{u}}_{\mathcal{U}})$ is the relative positive superdivisor of degree g of $\mathcal{U} \times \mathcal{V} \to \mathcal{V}$ defined by the equation

$$z^g - (a_1 + \theta \cdot b_1) z^{g-1} + \cdots + (-1)^g (a_g + \theta \cdot b_g)\,.$$

On the other hand, this is the characteristic polynomial of $z \otimes 1$ acting by multiplication on the structure ring of $\mathcal{Z}_{\mathcal{V}}$, so that this polynomial vanishes on $\mathcal{Z}_{\mathcal{V}}$, which means that $\mathcal{Z}_{\mathcal{V}}$ is contained in $(1 \times f)^*_{\mathcal{V}}(\mathcal{Z}^{\mathrm{u}}_{\mathcal{U}})$. Since both positive superdivisors have the same degree, they are equal, thus finishing the proof. □

4.5. THE CASE OF SUSY-CURVES

If $\mathcal{X}$ is a SUSY-curve, there exists an isomorphism $\psi\colon \mathcal{X} \xrightarrow{\sim} \mathcal{X}^{\mathrm{c}}$ between X and the supercurve of positive superdivisors of degree 1, as we proved in subsection 4.1. Then we have an isomorphism $S^g\mathcal{X} \to S^g\mathcal{X}^{\mathrm{c}}$ between the supersymmetric product of $\mathcal{X}$ and the superscheme $S^g\mathcal{X}^{\mathrm{c}}$ of positive superdivisors of degree g on X, so that the representability theorem now reads (see ref. [8]):

Theorem 7. *Let $\mathcal{X}$ be a SUSY-curve. The supersymmetric product $S^g\mathcal{X}$ represents the functor of positive superdivisors on $\mathcal{X}$, that is, there exists a universal relative* *positive superdivisor* $\mathcal{Z}_g^{\mathrm{u}}$ *of degree* g *of* $\mathcal{X} \times S^g\mathcal{X} \to S^g\mathcal{X}$ *such that the natural map*

$$\phi\colon \operatorname{Hom}(\mathcal{S}, S^g\mathcal{X}) \to \operatorname{Div}_{\mathcal{S}}^g(\mathcal{X} \times \mathcal{S}),$$
$$f \mapsto (1 \times f)^* \mathcal{Z}_g^{\mathrm{u}},$$

is a functorial isomorphism for every superscheme $\mathcal{S}$.

Moreover, since $1 \times \psi\colon \mathcal{X} \times \mathcal{X} \simeq \mathcal{X} \times \mathcal{X}^{\mathrm{c}}$ transforms by inverse image the universal positive superdivisor of degree 1 into Manin's superdiagonal, the universal superdivisor of $\mathcal{X} \times S^g\mathcal{X} \to S^g\mathcal{X}$ for SUSY-curves is constructed as in lemma 4 with Manin's superdiagonal playing the role of $\mathcal{Z}_1^{\mathrm{u}}$.

Summarizing, only for SUSY-curves, "unordered families of g points" (the points of $S^g\mathcal{X}$) are equivalent to "superdivisors of degree g" (the points of $S^g\mathcal{X}^{\mathrm{c}}$).

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